Journal of Applied Mathematics and Mechanics 72 (2008) 669–672

Contents lists available at ScienceDirect

Journal of Applied Mathematics and Mechanics

journal homepage: www.elsevier.com/locate/jappmathmech

The pursuit–evasion problem in a discrete version of the hamstrung car game☆

M.B. Ruziboyev

Tashkent, Uzbekistan

ARTICLE INFO

Article history:
Received 25 October 2007

ABSTRACT

A discrete version of the 'hamstrung car' game is considered. A triangle is taken as the terminal set. By a simple method, not requiring the solution of the quality problem, a sufficient evasion condition is obtained. ©Elsevier Ltd. All rights reserved.

1. Formulation of the problem

We will consider a discrete game[1–4] in R^{d} space, described by the following system of equations of motion

$$z(n+1) = F(z(n), u, \nu), \quad n \in N = \{0, 1, 2, \ldots\}, \quad u \in P, \quad \nu \in Q \tag{1.1}$$

Here u is the control parameter of the pursuer, ν is the control parameter of the evader, P and Q are non-empty compact subsets of spaces R^p and R^q, respectively, and $F{:}R^d \times P \times Q \to R^d$ is the specified continuous mapping. Below, a non-empty terminal set M is singled out in R^d.

Each sequence $u(\cdot)$: $N \to P$ is called a pursuer control, and a mapping of the type $V{:}N \times R^d \to Q$ is called an evader strategy. If the initial point z_0 is specified, the pursuer chooses strategy V and the evader chooses control $u(\cdot)$. Then the set $(z_0, u(\cdot), V)$ clearly generates the trajectory $z(\cdot)$: $N \to R^d$, which is defined by the recurrence formula

$$z(n+1) = F(z(n), u(n), V(n, z(n))), \quad z(0) = z_0$$

(Hence, in Eq. (1.1), $u = u(n)$ and $\nu = V(n, z(n))$).

By definition, from the point $z_0 \in R^d$ evasion is possible if an evasion strategy V exists such that, for any pursuer control $u(\cdot)$, the trajectory $z(\cdot)$ generated by $(z_0, u(\cdot), V)$ does not fall on the terminal set, i.e. $z(n) \notin M$ for all $n \in N$. If evasion is possible from each point $z_0 \in R^{\mathrm{d}}/M$, then we say that, in game (1.1), evasion is possible.[1,3,5]

We will assume that the terminal set M is a convex polyhedral set described by the relations

$$\langle \beta_i, z\rangle \le \alpha_i, \quad i = 1, 2, \ldots, m \tag{1.2}$$

where β_i are non-zero vectors, α_i are specified numbers and $\langle\cdot,\cdot\rangle$ is a scalar product. The following assertion has been proved (Ref.,[5] the corollary of Theorem 1_j with $j = 1$).

Theorem 1. For evasion in the discrete game (1.1) to be possible, it is sufficient for the condition $\bar{N} \subset M$ to be satisfied, where

$$\bar{N} = \{z \,|\, \max_{\nu, i}\min_{u}[\langle \beta_i, F(z, u, \nu)\rangle - \alpha_i] \le 0\}, \quad \nu \in Q, \quad u \in P \tag{1.3}$$

This easily proved assertion is entirely effective. For example, suppose the game (1.1) is linear, i.e.; $F(z, u, \nu) = Az - u + \nu$. In this case, the set $\bar{N}$ will also be a convex polyhedral set:

$$\bar{N} = \bigcap_{i=1}^{m} \{z \,|\, \langle A^*\beta_i, z\rangle - \max_u \langle \beta_i, u\rangle + \max_\nu \langle \beta_i, \nu\rangle \le \alpha_i\}$$

☆ *Prikl. Mat. Mekh.*, Vol. 72, No. 6, pp. 925–929, 2008.

E-mail address: m.ruziboev@yahoo.com.

0021-8928/$ – see front matter
doi:10.1016/j.jappmathmech.2009.01.002

where A^* is a transposed matrix. Therefore, checking the condition $\bar{N} \subset M$ reduces to the problem of the theory of inequalities,[6] which is solved in a finite number of steps. Below, using criterion (1.3), the condition is derived for possible evasion in a discrete version of the "hamstrung car" game, the equation of motion of which is non-linear (Ref.,[7] p. 46).

2. Derivation of the condition for evasion

In the "hamstrung car" game, the pursuer moves in the R^2 plane with maximum speed ρ, and the curvature of its trajectory should not exceed the magnitude r^{-1}, $r > 0$. The speed of motion of the evader (it is assumed that it is inertialess) does not exceed σ. In reduced coordinates, the game is described by the system (Ref.,[7] p. 48).

$$\dot{x} = -\rho r^{-1} u y + \upsilon_1 \sigma, \quad \dot{y} = \rho r^{-1} u x + \upsilon_2 \sigma - \rho; \quad |u| \le 1, \quad \sqrt{\nu_1^2 + \nu_2^2} \le 1 \tag{2.1}$$

where u is the control parameter of the pursuer and $\nu = (\upsilon_1, \upsilon_2)$ is the control parameter (velocity vector) of the evader. The values of these parameters must satisfy the constraints given above, and the condition for the game to be completed in the continuous case has the form

$$x^2 + y^2 \le l^2 \tag{2.2}$$

where l is the capture radius, $l > 0$.

It has been asserted (Ref.,[7] p. 293) that, if the inequality

$$l/r < \sqrt{1-\gamma^2} + \gamma \arcsin\gamma - 1 \tag{2.3}$$

is satisfied, where $\gamma = \sigma/\rho$, then the set of phase points (the evasion zone) from which completion of pursuit is possible consists of the circle (2.2) and a curvilinear triangle joined to it, and it was concluded that inequality (2.3) is the evasion condition. There is no reason to doubt this conclusion, but it must be pointed out that condition (2.3) was obtained using the heuristic method of characteristics, the application of which requires strict substantiation, which is far from simple (see Ref.).[8] Furthermore, condition (2.3) is derived by refined analytical calculations. Below, a sufficient condition for evasion in the discrete analogue of the "hamstrung car" game is obtained using Theorem 1 given above.

Replacement of the derivative dx/dt by the relation $(x(n+1)-x(n))/h$, $h > O$, Q, converts the system of differential Eq. (2.1) into a discrete system in the form of (1.1), namely.,

$$x(n+1) = x(n) - r^{-1} h\rho u y(n) + \nu_1 h\sigma, \quad y(n+1) = y(n) + r^{-1} h\rho u x(n) + \nu_2 h\sigma - h\rho \tag{2.4}$$

This system will be investigated on condition that the above constraints imposed on the control parameters n and ν are satisfied, and with the terminal set replaced by the triangle

$$M = \{(x, y) | y \ge 0, y + kx \le kl, y - kx \le kl\}, \quad k > 0 \tag{2.5}$$

(This replacement is an attempt to bring the set M as close as possible to the zone of capture in the differential game (see Ref.,[7] Fig. 9.1.2c)).

Note that earlier[5] the set M was replaced by the square $\{(x,y)$: $-l \le x,y \le l\}$, as a result of which the evasion condition obtained was ineffective, since, for fairly small h, evasion is only possible if $\sigma \ge \rho$. Below, we will assume that $\gamma = \sigma/\rho < 1$, which is natural for the evasion problem.

Lemma. We will assume that

$$k^2/\sqrt{k^2+1} \ge 1 \tag{2.6}$$

(or, which is equivalent, $k^2 \ge (1+\sqrt{5})/2$). A positive number h_0 then exists such that, when $h \in (0, h_0)$, the condition

$$l/r < (\gamma\sqrt{k^2+1} - 1)/k^2 \tag{2.7}$$

is sufficient for possible evasion in the game considered.

Note that the parameter k participants in condition (2.7). The evasion condition will be best if the maximum is taken on the right-hand side of condition (2.7) with respect to k, $k^2 \ge (1+\sqrt{5})/2$. Suppose $\gamma_0 = 2/\sqrt{5}$. Then, this maximum is equal to $(1-\sqrt{1-\gamma^2})/2$ if $\gamma \le \gamma_0$, and equal to $\gamma - (\sqrt{5}-1)/2$ if $\gamma > \gamma_0$. Thereby we have the following theorem.

Theorem 2. The condition

$$l/R < \begin{cases} (1-\sqrt{1-\gamma^2})/2, & \text{if } \gamma \le \gamma_0 \\ \gamma - (\sqrt{5}-1)/2, & \text{if } \gamma > \gamma_0 \end{cases} \tag{2.8}$$

is sufficient for possible evasion in game (2.4), (2.5) when $h \in (0, h_0)$.

Let us compare condition (2.8) with the Isaacs criterion (2.3). Fig. 1 shows graphs of the left-hand sides of inequality (2.3) (the dashed curves) and inequality (2.8) (the continuous curves) against the ratio $\gamma = \sigma/\rho$. It can be seen that, at low values of the ratio l/r, the range of γ values satisfying condition (2.8) is fairles close to the range of values from condition (2.3).

Proof of the lemma. We will construct in implicit form the set $\bar{N}$ corresponding to the discrete game considered. In the given case, the set M is a triangle defined by formula (2.5), so that (see inequality (1.2))

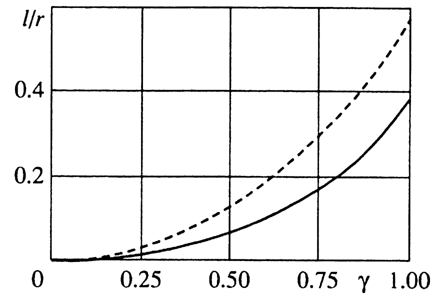

Fig. 1.

$$\beta_0 = (0, -1), \quad \beta_\varepsilon = (\varepsilon k, 1), \quad \alpha_0 = 0, \quad \alpha_\varepsilon = kl; \quad \varepsilon = \pm 1$$

The inequality

$$\max_\nu \min_u \langle \beta_\varepsilon, F(z, u, \nu) \rangle - \alpha_\varepsilon \le 0$$

has the form

$$-y - h\rho r^{-1}|x| + h(\rho + \sigma) \le 0 \quad \text{при} \quad \varepsilon = 0$$

$$\varepsilon kx + y - h\rho r^{-1}|x - \varepsilon ky| - W \le 0 \quad \text{при} \quad \varepsilon = \pm 1; \quad W = h\rho + kl - h\sigma\sqrt{k^2 + 1} \tag{2.9}$$

Thus, the set $\bar{N}$ is defined by relations (2.9), i.e., N is a polygon bounded by the lines L_0 and L_ε, specified respectively by the equations

$$-y - h\rho r^{-1}|x| + h(\rho + \sigma) = 0 \quad \text{и} \quad \varepsilon kx + y - h\rho r^{-1}|x - \varepsilon ky| - W = 0$$

where $\varepsilon = \pm 1$.

Elementary calculations indicate that $\bar{N}$ is a non-convex pentagon (see Fig. 2) with vertices at the points $A(0,a)$, $B_\varepsilon(\varepsilon b, b/k)$, $C_\varepsilon(\varepsilon x_c, y_c)$ and $D(0, d)$, where

$$a = Wr/(r - kh\rho), \quad b = rkh(\rho + \sigma)/(r + kh\rho), \quad d = h(\rho + \sigma)$$

$$x_c = r(-kh^2\rho(\rho + \sigma) + rkl - rh\sigma(1 + \sqrt{k^2 + 1}))S^{-1}$$

$$y_c = rh(rk(\rho + \sigma) - 2h\rho^2 - kl\rho - h\rho\sigma(1 - \sqrt{k^2 + 1}))S^{-1}$$

$$S = r^2k - 2rh\rho - kh^2\rho^2$$

We will derive the conditions which ensure the inclusion $\bar{N} \subset M$, taking into account that $h < (2\rho)^{-1}r$. For the inclusion $\bar{N} \subset M$ to occur, it is sufficient to check the conditions $A \in M$ and $C_\varepsilon \in M$, $\varepsilon = \pm 1$. (It is possible to ascertain that the points B_{+1}, B_{-1} and D lie within the triangle $C_{-1}AC_{+1}$. Below, taking into account the symmetry of the entire picture about to the ordinate axis, we will confine ourselves to considering of the case $\varepsilon = \pm 1$.

The condition $A \in M$ is ensured by the inequality $a = Wr/(r - kh\rho) \le kl$, which, in turn, is equivalent to condition (2.7).

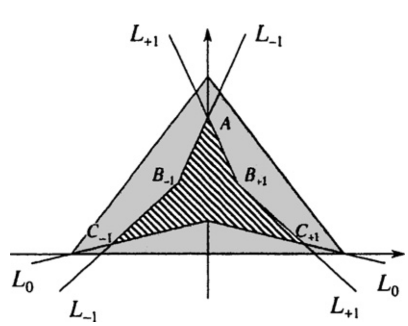

Fig. 2.

In order to check the condition $C_{+1} \in M$, we will present the coordinates of point C_{+1} in the form

$$x_c = \bar{x}_c + O(h^2), \quad y_c = \bar{y}_c + O(h^2)$$

$$\bar{x}_c = \frac{r(kl - h\sigma(1 + \sqrt{k^2+1}))}{rk - 2h\rho}, \quad \bar{y}_c = \frac{kh(r(\rho+\sigma) - l\rho)}{rk - 2h\rho}$$

(Note that, from the conditions $h < (2k\rho)^{-1} r$ and $k^2 > (1+\sqrt{5})/2$, it follows that $rk > 2h\rho$).

An h_0, $h_0 > 0$ exists, such that, when $h \in (0, h_0)$, the condition $(\bar{x}_c, \bar{y}_c) \in \operatorname{int} M$ implies the required relation $C_{+1} = (x_c, y_c) \in M$, i.e., $C_{+1} \in M$ (int M denotes the interior of M). The condition $(\bar{x}_c, \bar{y}_c) \in \operatorname{int} M$ is equivalent to the system of inequalities $\bar{y}_c > 0$, $k\bar{x}_c + \bar{y}_c < kl$, which are transformed into

$$l/r < 1 + \gamma, \quad l/r < \gamma\sqrt{k^2+1} - 1 \tag{2.10}$$

By virtue of condition (2.6), inequalities (2.10) are simple corollaries of condition (2.7).

Acknowledgements

I wish to thank A. A. Azamov and A. Sh. Kuchkarov for helpful discussions.

References

1. Pontryagin LS, Mishchenko YeF. The evasion problem in linear differential games. *Differents Uravneniya* 1971;**7**(3):436–45.
2. Satimov NYu. The evasion problem for a class of non-linear discrete games. *Izv Akad Nauk SSSR Tekhn Kibernetika* 1973;**6**:45–8.
3. Chikrii AA. Linear discrete games of quality. *Kibernetika* 1971;(5):90–9.
4. Baston VJ, Bostok FA. Discrete hamstrung squad car games. *Intern J Game Theory* 1987;**16**(4):253–61.
5. Satimov N, Azamov A. Non-linear discrete pursuit-evasion games. *Kibernetika* 1976;(4):70–4.
6. Chernikov SN. *Linear Inequalities*. Moscow: Nauka; 1968.
7. Isaacs R. *Differential Games*. New York: Wiley; 1965.
8. Pontryagin LS. The theory of Differential games. *Uspekhi Mat Nauk* 1966;**21**(4):219–74.

Translated by P.S.C.